EPYMT TDG Group 2 Problem Set 1

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09-08-2024

1 Integration

Compute the following integrals:

i $\displaystyle\int \frac{x}{x^2+2x+4}\,dx$

ii $\displaystyle\int \frac{3x^2-2x}{1+x^2}\,dx$

iii $\displaystyle\int \frac{1}{2\cos 1.5x}\,dx$

iv (1) $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2 t}\,dt.$

(2) $\displaystyle\int_0^{\pi} \frac{\cos t}{1+\sin^2 t}\,dt.$

(3) $\displaystyle\int_0^{\pi} \frac{\sin 2t}{1+\sin^2 t}\,dt.$

v (Partial fraction decomposition)

(1) If $\dfrac{1}{(x-a)(x-b)} = \dfrac{C}{x-a} + \dfrac{D}{x-b}$ for some constants C, D, find C and D.

(2) Using (1), evaluate the following integrals:

(a) $\displaystyle\int_0^{1} \frac{1}{3x^2+10x+3}\,dx.$

(b) $\displaystyle\int_0^{2} \frac{x}{2x^2-4x-6}\,dx.$

(3) With ideas from (1), try to compute $\displaystyle\int \frac{1}{x^3-2x^2-x+2}\,dx$

vi $\displaystyle\int_1^{2} \frac{1}{e^x-e^{-x}}\,dx.$

2 Matrix

Find the inverse of the following matrices by elementary row operation:

i $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 5 \\ 4 & 2 & 2 \end{pmatrix}$

ii $\begin{pmatrix} 2 & 0 & 2 \\ 9 & 5 & 2 \\ 4 & 3 & 0 \end{pmatrix}$

3 Basis and linear independence

i Determine whether the following collection of vectors forms is linearly independent. Also determine whether it forms a basis of $\mathbb{R}^3$:

(1) $\{(1,2,3),(4,5,6),(7,8,9)\}$

(2) $\{(1,0,0),(0,2,2)\}$

(3) $\{(1,2,4),(0,0,2),(2,2,0)\}$

ii Determine whether the following statement is correct or not:
Let $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$ such that $\{\mathbf{a},\mathbf{b}\}$, $\{\mathbf{b},\mathbf{c}\}$ and $\{\mathbf{c},\mathbf{a}\}$ are linearly independent sets. Then $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a linearly independent set.

iii Let $\mathbf{u},\mathbf{v},\mathbf{w}$ are distint vectors in $\mathbb{R}^4$. Prove the following statements:

(1) $\{\mathbf{u},\mathbf{v}\}$ is a linearly independent set if and only if $\{\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}\}$ is linearly independent set.

(2) $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a linearly independent set if and only if $\{\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{w},\mathbf{v}+\mathbf{w}\}$ is linearly independent set.

4 Linear Transformation

i Find the matrix representation of the following linear transformation:

(1) Reflecting a vector in $\mathbb{R}^2$ by $y = ax$, where $a > 0$

(2) Rotating a vector in $\mathbb{R}^3$ by 30 degrees clockwisely about the $z-$axis.

(3) Multiply the $x-$coordinate of a vector in $\mathbb{R}^3$ by 3.

ii Can we find a 2×2 matrix representation of the translation of a vectors in $\mathbb{R}^2$? If yes, write down the matrix. If no, explain why.

5 Others

Please finish those exercises provided in the tutorial notes 1.

The following question aims to provide a way to show that a differentiable function $f'(x) = 0$ on an interval (a,b) is constant:

Recall that the definition of derivative is

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

and f attains a local minimum at c we have for $f(x) \geq f(c)$ for all $x \in (c-\delta, c+\delta)$ for some $\delta > 0$. f attains a local maximum at c we have for $f(x) \leq f(c)$ for all $x \in (c-\delta, c+\delta)$ for some $\delta > 0$.

i Show that if f attains local maximum or local minimum in $c \in (a,b)$ and f is differentiable at c, then we have $f'(c) = 0$.

ii Do we have the same result if f attains local maximum or minimum at a or b, with f differentiable at the end-points? Besides, is the converse of (a) true?

iii (Try it only if you want some challenge. It's ok to skip it.) (Mean Value Theorem) Prove that there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

You can use the following theorem without proof:

Theorem 5.1 (Rolle's theorem)**.** *If $f(a) = f(b) = 0$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

(The proof of this theorem requires (a) and some theorems on continuous functions (called the extreme value theorem), which we will not discuss here.)

iv Hence, show that if f is continuous on $[a,b]$ and $f'(x) = 0$ on (a,b), then f is constant on $[a,b]$. (We will use this skills in the future.)